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The polarization vector and secular equation for surface waves in an anisotropic elastic half-space

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This paper is dedicated to Professor Bruno Boley

Abstract

The displacement at the free surface of an anisotropic elastic half-space $x_2 > 0$ generated by a surface wave propagating in the direction of the x_1-axis traces an elliptic path. It is represented by the polarization vector $\mathbf{a}_R = \mathbf{e}_1 + i\mathbf{e}_2$, where $\mathbf{e}_1$, $\mathbf{e}_2$ are the conjugate radii of the ellipse on the polarization plane. The displacement traces the ellipse in the direction from $\mathbf{e}_1$ to $\mathbf{e}_2$. We present explicit expressions of $\mathbf{e}_1$, $\mathbf{e}_2$ and the secular equation without computing the Stroh eigenvalues p and the associated eigenvectors $\mathbf{a}$ and $\mathbf{b}$. After presenting the expressions for a general anisotropic elastic material, the special cases are studied separately. For monoclinic materials with the symmetry plane at $x_3 = 0$, the secular equation and the conjugate radii $\mathbf{e}_1$, $\mathbf{e}_2$ are identical to that for orthotropic materials when $s'_{16} = 0$ but s'_{26} need not vanish. For monoclinic materials with the symmetry plane at $x_1 = 0$, $\mathbf{e}_1$ is along the x_1-axis while $\mathbf{e}_2$ is on the plane $x_1 = 0$. If the symmetry plane is at $x_2 = 0$, $\mathbf{e}_1$ is on the plane $x_2 = 0$ while $\mathbf{e}_2$ is along the negative x_2-axis. In both cases, $\mathbf{e}_1$, $\mathbf{e}_2$ are the principal radii of the ellipse. We also present the derivative of $\mathbf{a}_R$ with respect to x_2, the depth from the free surface, that provides information on (i) whether the conjugate radii of the ellipse increase as x_2 increases and (ii) whether the polarization plane rotates as x_2 increases. New secular equations are obtained for monoclinic materials with the symmetry plane at $x_1 = 0$ or $x_2 = 0$.

Keywords: Surface waves; Secular equations; Polarization vector; Anisotropic materials

1. Introduction

A surface wave propagating in the direction of the x_1-axis in an anisotropic elastic half-space $x_2 > 0$ in general consists of two or three inhomogeneous partial waves. The displacement $\mathbf{u}$ is expressed in terms of the Stroh eigenvalues p_α and the associated eigenvectors $\mathbf{a}_\alpha$ ($\alpha = 1, 2, 3$). The stress is expressed in terms of another eigenvectors $\mathbf{b}_\alpha$ ($\alpha = 1, 2, 3$). The vanishing of the surface traction at the free surface is achieved by

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finding a proper linear superposition of $\mathbf{b}_\alpha$. The same linear superposition of $\mathbf{a}_\alpha$ provides the polarization vector $\mathbf{a}_R$ at the free surface. With this approach, an explicit expression of $\mathbf{a}_R$ can be obtained only for special anisotropic materials.

We present a set of equations that govern the polarization vector $\mathbf{a}_R$. The equations were originally derived by Taziev (1989) but he was interested mainly in obtaining an explicit expression for secular equation. He did have the expression for the polarization vector but it was not explicit. What prevented him from obtaining an explicit expression for $\mathbf{a}_R$ was that explicit expressions of the matrices $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ were not available in his time. He tried to employ the result for a general anisotropic elastic material to special materials. Examples in the literature show that this is very often not the best way to study surface waves for special materials.

The Stroh (1958, 1962) formalism is introduced briefly in Section 2 which is employed in Section 3 to derive the secular equation obtained by Taziev (1989). An explicit expression of the polarization vector $\mathbf{a}_R$ for a general anisotropic elastic material is presented in Section 4. Also presented is the derivative of $\mathbf{a}_R$ with respect to x_2. The secular equation and the polarization vector and its derivative for special materials are best studied separately instead of specializing the general result. Thus the special cases of orthotropic and monoclinic materials with the symmetry plane at $x_3 = 0$, $x_1 = 0$ or $x_2 = 0$ are discussed separately in Sections 5–8. New secular equations are obtained for monoclinic materials with the symmetry plane at $x_1 = 0$ or $x_2 = 0$. It is shown in Section 9 that the derivation presented here applies to two- and one-component surface waves as well so that the results are valid for both subsonic and supersonic surface waves.

2. The Stroh formalism

In a fixed rectangular coordinate system x_i $(i = 1, 2, 3)$ let u_i and σ_{ij} be the displacement and stress in an anisotropic elastic material. The stress–strain law and the equation of motion are

$$\sigma_{ij} = C_{ijks} u_{k,s}, \tag{2.1}$$

$$C_{ijks} u_{k,sj} = \rho \ddot{u}_i, \tag{2.2}$$

in which repeated indices imply summation, ρ is the mass density, the comma denotes differentiation with x_i, the dot stands for differentiation with time t, and C_{ijks} is the elastic stiffness that is assumed to possess the full symmetry. Consider an inhomogeneous plane wave with the steady wave speed v propagating in the direction of the x_1-axis. A solution for the displacement vector $\mathbf{u}$ of Eq. (2.2) can be written as (Stroh, 1962)

$$\mathbf{u} = \mathbf{a} e^{ikz}, \quad z = x_1 - vt + px_2, \tag{2.3}$$

in which k is the real wave number, and p and $\mathbf{a}$ satisfy the eigenrelation

$$\Gamma \mathbf{a} = \mathbf{0}, \tag{2.4}$$

$$\Gamma = \mathbf{Q} - X\mathbf{I} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2 \mathbf{T}, \tag{2.5}$$

$$X = \rho v^2. \tag{2.6}$$

In the above the superscript T stands for the transpose, $\mathbf{I}$ is the unit matrix, and $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are 3×3 matrices whose elements are

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{2.7a}$$

In the contracted notation $C_{\alpha\beta}$ they are

$$\mathbf{Q}=\begin{bmatrix}C_{11} & C_{16} & C_{15}\\ C_{61} & C_{66} & C_{65}\\ C_{51} & C_{56} & C_{55}\end{bmatrix},\quad \mathbf{R}=\begin{bmatrix}C_{16} & C_{12} & C_{14}\\ C_{66} & C_{62} & C_{64}\\ C_{56} & C_{52} & C_{54}\end{bmatrix},\quad \mathbf{T}=\begin{bmatrix}C_{66} & C_{62} & C_{64}\\ C_{26} & C_{22} & C_{24}\\ C_{46} & C_{42} & C_{44}\end{bmatrix}. \tag{2.7b}$$

The matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and positive definite. Introducing the new vector $\mathbf{b}$ defined by

$$\mathbf{b}=(\mathbf{R}^{\mathrm{T}}+p\mathbf{T})\mathbf{a}=-[p^{-1}(\mathbf{Q}-X\mathbf{I})+\mathbf{R}]\mathbf{a} \tag{2.8}$$

in which the second equality follows from (2.4), the stress determined from (2.1) can be written as

$$\sigma_{i1}=-\phi_{i,2}-\rho\upsilon\dot{u}_i,\quad \sigma_{i2}=\phi_{i,1}. \tag{2.9}$$

The ϕ_i $(i=1,2,3)$ are the components of the stress function vector

$$\boldsymbol{\phi}=\mathbf{b}\mathrm{e}^{\mathrm{i}kz}. \tag{2.10}$$

There are six eigenvalues p_α and six Stroh eigenvectors $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ $(\alpha=1,2,\ldots,6)$. When p_α are complex they consist of three pairs of complex conjugates. If p_1, p_2, p_3 are the eigenvalues with a positive imaginary part, the remaining three eigenvalues are the complex conjugates of p_1, p_2, p_3. Assuming that p_1, p_2, p_3 are distinct, the general solution obtained from superposing three solutions of (2.3) and (2.10) associated with p_1, p_2, p_3 can be written in matrix notation as

$$\mathbf{u}=\mathbf{A}\langle\mathrm{e}^{\mathrm{i}kz_*}\rangle\mathbf{q},\quad \boldsymbol{\phi}=\mathbf{B}\langle\mathrm{e}^{\mathrm{i}kz_*}\rangle\mathbf{q}, \tag{2.11}$$

where $\mathbf{q}$ is an arbitrary constant vector and

$$\mathbf{A}=[\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3],\quad \mathbf{B}=[\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3], \tag{2.12a}$$

$$\langle\mathrm{e}^{\mathrm{i}kz_*}\rangle=\mathrm{diag}[\mathrm{e}^{\mathrm{i}kz_1},\ \mathrm{e}^{\mathrm{i}kz_2},\ \mathrm{e}^{\mathrm{i}kz_3}], \tag{2.12b}$$

$$z_\alpha=x_1-\upsilon t+p_\alpha x_2. \tag{2.12c}$$

For surface waves in the half-space $x_2\geqslant 0$, $(2.11)_1$ assures us that $\mathbf{u}\to\mathbf{0}$ as $x_2\to\infty$. The surface traction at $x_2=0$ vanishes if $\boldsymbol{\phi}=\mathbf{0}$ at $x_2=0$, i.e.,

$$\mathbf{Bq}=\mathbf{0}. \tag{2.13}$$

Thus when υ is the surface wave speed the determinant of $\mathbf{B}$ must vanish,

$$|\mathbf{B}|=0. \tag{2.14}$$

This is the secular equation for X.

The two equations in (2.8) can be written in a standard eigenrelation as (Ingebrigtsen and Tonning, 1969; Barnett and Lothe, 1973; Chadwick and Smith, 1977)

$$\mathbf{N}\xi=p\xi, \tag{2.15}$$

$$\mathbf{N}=\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3+X\mathbf{I} & \mathbf{N}_1^{\mathrm{T}}\end{bmatrix},\quad \xi=\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}, \tag{2.16}$$

$$\mathbf{N}_1=-\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}},\quad \mathbf{N}_2=\mathbf{T}^{-1},\quad \mathbf{N}_3=\mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}-\mathbf{Q}. \tag{2.17}$$

The matrix $\mathbf{N}_2$ is symmetric and positive definite while $\mathbf{N}_3$ is symmetric and positive semi-definite. It is shown in Ting (1988) that $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ have the structures

$$-\mathbf{N}_1=\begin{bmatrix}r_6 & 1 & s_6\\ r_2 & 0 & s_2\\ r_4 & 0 & s_4\end{bmatrix},\quad \mathbf{N}_2=\begin{bmatrix}n_{66} & n_{26} & n_{46}\\ n_{26} & n_{22} & n_{24}\\ n_{46} & n_{24} & n_{44}\end{bmatrix},\quad -\mathbf{N}_3=\begin{bmatrix}\eta & 0 & -\kappa\\ 0 & 0 & 0\\ -\kappa & 0 & \mu\end{bmatrix}. \tag{2.18}$$

An explicit expression of the elements of $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ is given in Ting (1988) in terms of the reduced elastic compliances $s'_{\alpha\beta}$ as (see also Ting, 1996, p. 167)

$$\begin{aligned}&\mu = s'_{11}/\Delta, \quad \eta = s'_{55}/\Delta, \quad \kappa = s'_{15}/\Delta, \quad n_{\alpha\beta} = s'(\alpha,1,5|\beta,1,5)/\Delta, \quad r_\alpha = s'(1,5|5,\alpha)/\Delta,\\ &s_\alpha = s'(1,5|\alpha,1)/\Delta, \quad \Delta = s'(1,5) > 0.\end{aligned} \tag{2.19}$$

In the above, $s'(n_1,\dots,n_k|m_1,\dots,m_k)$ is the $k\times k$ minor of the matrix $s'_{\alpha\beta}$, the elements of which belong to the rows of $s'_{\alpha\beta}$, numbered $n_1,\dots,n_k$ and columns numbered $m_1,\dots,m_k$, $1\leqslant k\leqslant 6$. A principal minor is $s'(n_1,\dots,n_k|n_1,\dots,n_k)$, which is written as $s'(n_1,\dots,n_k)$ for simplicity. Removing the third row and the third column of $s'_{\alpha\beta}$ that contain only zero elements, the 5×5 matrix is positive definite. Hence $\Delta > 0$. μ is the shear modulus when the material is isotropic. An explicit expression of $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ in terms of the elastic stiffnesses $C_{\alpha\beta}$ is given in Barnett and Chadwick (1990).

3. Equations for the polarization vector

At the free surface $x_2 = 0$, $(2.11)_1$ gives

$$\mathbf{u}(x_1,0,t) = \mathbf{a}_R e^{ik(x_1-vt)}, \quad \mathbf{a}_R = \mathbf{A}\mathbf{q}, \tag{3.1}$$

where $\mathbf{a}_R$ is the polarization vector of the surface waves at the free boundary. Eq. (2.15) consists of two equations,

$$\begin{aligned}&\mathbf{N}_1\mathbf{a} + \mathbf{N}_2\mathbf{b} = p\mathbf{a},\\ &(\mathbf{N}_3 + X\mathbf{I})\mathbf{a} + \mathbf{N}_1^{\mathrm{T}}\mathbf{b} = p\mathbf{b}.\end{aligned} \tag{3.2}$$

A linear superposition of three equations obtained from $(3.2)_2$ with $p = p_1, p_2, p_3$ leads to

$$(\mathbf{N}_3 + X\mathbf{I})\mathbf{A}\mathbf{q} + \mathbf{N}_1^{\mathrm{T}}\mathbf{B}\mathbf{q} = \mathbf{B}\langle p_*\rangle\mathbf{q}. \tag{3.3}$$

The six-vectors $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ and $(\mathbf{b}_\beta, \mathbf{a}_\beta)$ are orthogonal to each other when $p_\alpha \neq p_\beta$. The orthogonality relation can be written as (Barnett and Lothe, 1973; see also Ting, 1996, p. 146)

$$\overline{\mathbf{B}}^{\mathrm{T}}\mathbf{A} + \overline{\mathbf{A}}^{\mathrm{T}}\mathbf{B} = \mathbf{0}. \tag{3.4}$$

Pre-multiplying (3.3) by $\bar{\mathbf{q}}^{\mathrm{T}}\overline{\mathbf{A}}^{\mathrm{T}}$ and using (3.4), $(3.1)_2$, (2.13), we obtain

$$\bar{\mathbf{a}}_R^{\mathrm{T}}(\mathbf{N}_3 + X\mathbf{I})\mathbf{a}_R = 0. \tag{3.5}$$

Hence X is bounded by the largest and smallest eigenvalues of $-\mathbf{N}_3$ (Ting, 1996, p. 472). When $X = 0$, (3.5) recovers the identity obtained by Stroh (1958).

Eq. (3.5) can be generalized. First we generalize (2.15) as

$$\mathbf{N}^n\boldsymbol{\xi} = p^n\boldsymbol{\xi}, \tag{3.6}$$

where n is any positive or negative integer. Let

$$\mathbf{N}^n = \begin{bmatrix}\mathbf{N}_1^{(n)} & \mathbf{N}_2^{(n)}\\ \mathbf{K}^{(n)} & \mathbf{N}_1^{(n)\mathrm{T}}\end{bmatrix} \tag{3.7}$$

in which

$$\mathbf{N}_1^{(1)} = \mathbf{N}_1, \quad \mathbf{N}_2^{(1)} = \mathbf{N}_2, \quad \mathbf{K}^{(1)} = \mathbf{N}_3 + X\mathbf{I}. \tag{3.8}$$

For $n = -1$, it can be shown that (Ting, 1996, p. 451)

$$\mathbf{N}_1^{(-1)} = \mathbf{N}_2^{(-1)}\mathbf{R}, \quad \mathbf{N}_2^{(-1)} = -(\mathbf{Q} - X\mathbf{I})^{-1}, \quad \mathbf{K}^{(-1)} = \mathbf{T} + \mathbf{R}^{\mathrm{T}}\mathbf{N}_2^{(-1)}\mathbf{R}. \tag{3.9}$$

According to the Cayley–Hamilton theorem (Hohn, 1965), a matrix satisfies its own characteristic equation. Since $\mathbf{N}$ is a 6×6 matrix only five of $\mathbf{N}^n$ are independent. Other $\mathbf{N}^n$ can be expressed in terms of the five $\mathbf{N}^n$. Hence it suffices to consider $\mathbf{N}^n$ for five different n. Eq. (3.6) consists of two equations of which the second one is

$$\mathbf{K}^{(n)}\mathbf{a} + \mathbf{N}_1^{(n)^{\mathrm{T}}}\mathbf{b} = p^n\mathbf{b}. \tag{3.10}$$

Following the derivation of (3.5) from (3.2), we obtain from (3.10)

$$\bar{\mathbf{a}}_R^{\mathrm{T}}\mathbf{K}^{(n)}\mathbf{a}_R = 0. \tag{3.11}$$

Choosing any five n, we have the equations for the polarization vector $\mathbf{a}_R$. For $n = 1, 2, -2$, it recovers the identities (3.23) and $(3.24)_{1,2}$ in Currie (1979).

From $\mathbf{N}^{n\pm1} = \mathbf{N}^n\mathbf{N}^{\pm1} = \mathbf{N}^{\pm1}\mathbf{N}^n$ we obtain

$$\mathbf{N}_1^{(n\pm1)} = \mathbf{N}_1^{(n)}\mathbf{N}_1^{(\pm1)} + \mathbf{N}_2^{(n)}\mathbf{K}^{(\pm1)} = \mathbf{N}_1^{(\pm1)}\mathbf{N}_1^{(n)} + \mathbf{N}_2^{(\pm1)}\mathbf{K}^{(n)}, \tag{3.12a}$$

$$\mathbf{N}_2^{(n\pm1)} = \mathbf{N}_1^{(n)}\mathbf{N}_2^{(\pm1)} + \mathbf{N}_2^{(n)}\mathbf{N}_1^{(\pm1)^{\mathrm{T}}} = \mathbf{N}_1^{(\pm1)}\mathbf{N}_2^{(n)} + \mathbf{N}_2^{(\pm1)}\mathbf{N}_1^{(n)^{\mathrm{T}}}, \tag{3.12b}$$

$$\mathbf{K}^{(n\pm1)} = \mathbf{K}^{(n)}\mathbf{N}_1^{(\pm1)} + \mathbf{N}_1^{(n)^{\mathrm{T}}}\mathbf{K}^{(\pm1)} = \mathbf{K}^{(\pm1)}\mathbf{N}_1^{(n)} + \mathbf{N}_1^{(\pm1)^{\mathrm{T}}}\mathbf{K}^{(n)}. \tag{3.12c}$$

Adding the two expressions for $\mathbf{K}^{(n\pm1)}$ in (3.12c) leads to

$$2\mathbf{K}^{(n\pm1)} = [\mathbf{K}^{(n)}\mathbf{N}_1^{(\pm1)} + \mathbf{N}_1^{(\pm1)^{\mathrm{T}}}\mathbf{K}^{(n)}] + [\mathbf{K}^{(\pm1)}\mathbf{N}_1^{(n)} + \mathbf{N}_1^{(n)^{\mathrm{T}}}\mathbf{K}^{(\pm1)}]. \tag{3.13}$$

It tells us that if $\mathbf{K}^{(n)}$ is symmetric so is $\mathbf{K}^{(n\pm1)}$. Since $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(-1)}$ are symmetric, $\mathbf{K}^{(n)}$ is symmetric for all n. Likewise, it can be shown that $\mathbf{N}_2^{(n)}$ is symmetric for all n. The matrices $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ do not depend on X. From (3.8) and (3.12), the elements of $\mathbf{K}^{(n)}$ are polynomial in X of degree no more than one for $n = 1, 2$, no more than two for $n = 3, 4$, and no more than three for $n = 5$.

Assuming that $x_1 = 0$ is not a polarization plane, let

$$\mathbf{a}_R^{\mathrm{T}} = [1, \ \alpha, \ \beta]. \tag{3.14}$$

Eq. (3.11) can be written in full as

$$K_{11}^{(n)} + K_{22}^{(n)}\alpha\bar{\alpha} + K_{33}^{(n)}\beta\bar{\beta} + K_{12}^{(n)}(\alpha + \bar{\alpha}) + K_{13}^{(n)}(\beta + \bar{\beta}) + K_{23}^{(n)}(\alpha\bar{\beta} + \bar{\alpha}\beta) = 0. \tag{3.15}$$

Setting $n = 1$–5, (3.15) provides five equations that can be solved for, say,

$$\alpha + \bar{\alpha} = f_1, \quad \alpha\bar{\alpha} = f_2, \quad \beta + \bar{\beta} = f_3, \quad \beta\bar{\beta} = f_4, \quad \alpha\bar{\beta} + \bar{\alpha}\beta = f_5. \tag{3.16}$$

When α, $\bar{\alpha}$, β, $\bar{\beta}$ are computed from the first four equations and the results are inserted into the fifth equation in (3.16), Taziev (1989) obtained the secular equation

$$f_1^2f_4 + f_3^2f_2 + f_5^2 - f_1f_3f_5 - 4f_2f_4 = 0 \tag{3.17}$$

for a general anisotropic elastic material.

It is not known if there exists a case in which $x_1 = 0$ is a polarization plane. If it exists, both α and β computed from (3.16) would be unbounded. In this case, (3.14) is replaced by

$$\mathbf{a}_R^{\mathrm{T}} = [1/\alpha, \ 1, \ \beta/\alpha] \quad \text{or} \quad [1/\beta, \ \alpha/\beta, \ 1]. \tag{3.18}$$

It is known that $x_2 = 0$ is the polarization plane for one-component surface waves (see Section 8). Barnett (1992) has shown that $x_2 = 0$ cannot be a polarization plane for a subsonic two-component surface waves. The question is open if $x_2 = 0$ can be a polarization plane for three-component and supersonic surface waves.

The 3×3 symmetric matrices $\mathbf{K}^{(n)}$ can be computed recursively using $(3.12a)_2$ and (3.12c). The computation of $\mathbf{K}^{(n)}$ becomes more complicated for the higher orders $n = 4$ and 5. Instead of $n = 4$ and 5 we can replace them by $n = -1$ and -2. The $\mathbf{K}^{(n)}$ for monoclinic materials with the symmetry plane at $x_3 = 0$, $x_1 = 0$ or $x_2 = 0$ are presented in Appendix A.

4. The polarization vector $\mathbf{a}_R$

Taziev (1989) was interested mainly in the secular equation (3.17). He was not interested in the polarization vector even though the α, $\bar{\alpha}$, β, $\bar{\beta}$ computed from the first four equations in (3.16) provide the polarization vector $\mathbf{a}_R$ defined in (3.14). Let

$$\alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2, \tag{4.1}$$

where α_1, α_2, β_1, β_2 are real. The polarization vector (3.14) is

$$\mathbf{a}_R = \mathbf{e}_1 + i\mathbf{e}_2, \tag{4.2}$$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ \alpha_1 \\ \beta_1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ \alpha_2 \\ \beta_2 \end{bmatrix}. \tag{4.3}$$

Eqs. (3.15) and (3.16) are rewritten as

$$K_{11}^{(n)} + K_{22}^{(n)}(\alpha_1^2 + \alpha_2^2) + K_{33}^{(n)}(\beta_1^2 + \beta_2^2) + 2K_{12}^{(n)}\alpha_1 + 2K_{13}^{(n)}\beta_1 + 2K_{23}^{(n)}(\alpha_1\beta_1 + \alpha_2\beta_2) = 0. \tag{4.4}$$

$$2\alpha_1 = f_1, \quad \alpha_1^2 + \alpha_2^2 = f_2, \quad 2\beta_1 = f_3, \quad \beta_1^2 + \beta_2^2 = f_4, \quad 2(\alpha_1\beta_1 + \alpha_2\beta_2) = f_5. \tag{4.5}$$

Eq. (4.4) remains the same if α_2 and β_2 change the sign. In the special case of isotropic materials, it is known that $\alpha_1 = \beta_1 = \beta_2 = 0$ while $\alpha_2 < 0$. Hence we let

$$\alpha_2 < 0, \tag{4.6}$$

so that α_2 and β_2 are unique. We obtain from $(4.5)_{1,2}$,

$$\alpha_1 = f_1/2, \quad \alpha_2 = -\sqrt{f_2 - \alpha_1^2}. \tag{4.7}$$

To avoid nonuniqueness, it is best to compute β_1 and β_2 from $(4.5)_{3,5}$ as

$$\beta_1 = f_3/2, \quad \beta_2 = [(f_5/2) - \alpha_1\beta_1]/\alpha_2. \tag{4.8}$$

Substitution of (4.8) into $(4.5)_4$ recovers the secular equation (3.17).

The displacement at the free surface shown in (3.1) is

$$\mathbf{u}(x_1, 0, t) = (\mathbf{e}_1 + i\mathbf{e}_2)e^{ik(x_1 - vt)} = \hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2, \tag{4.9}$$

where

$$\hat{\mathbf{e}}_1 = \mathbf{e}_1 \cos\psi - \mathbf{e}_2 \sin\psi, \quad \hat{\mathbf{e}}_2 = \mathbf{e}_1 \sin\psi + \mathbf{e}_2 \cos\psi, \tag{4.10}$$

and $\psi = k(x_1 - vt)$. Since $\hat{\mathbf{e}}_1(\psi) = \hat{\mathbf{e}}_2(\psi + \pi/2)$, it does not matter if we take the real part $\hat{\mathbf{e}}_1$ or the imaginary part $\hat{\mathbf{e}}_2$ as the solution for the displacement at the free surface. As t increases (or ψ decreases), the displacement traces an ellipse in the direction from $\mathbf{e}_1$ to $\mathbf{e}_2$. A pair of diameters in an ellipse are said to be *conjugate* if all chords parallel to one diameter are bisected by the other diameter. Therefore the tangent to the ellipse at the extremity of one diameter is parallel to the other diameter. $\mathbf{e}_1$, $\mathbf{e}_2$ are a pair of *conjugate radii*, so are $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ (Gibbs, 1961; Synge, 1966; Stone, 1963; Boulanger and Hayes, 1991). When the conjugate radii are orthogonal to each other, they are the principal radii of the ellipse. The conjugate radii $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ are the principal radii when

$$\tan 2\psi = \frac{2\mathbf{e}_1 \cdot \mathbf{e}_2}{\mathbf{e}_2 \cdot \mathbf{e}_2 - \mathbf{e}_1 \cdot \mathbf{e}_1}. \tag{4.11}$$

It is readily shown from (4.10) that (Ting, 1996, p. 26)

$$\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = \mathbf{e}_1 \cdot \mathbf{e}_1 + \mathbf{e}_2 \cdot \mathbf{e}_2. \tag{4.12}$$

Thus the sum of the square of the lengths of the conjugate radii is independent of the choice of the conjugate radii, and hence is identical to the sum of the square of the principal radii.

We could also study how the polarization vector varies with respect to the depth at the free surface. When $(2.11)_1$ is differentiated with kx_2 and the result is evaluated at the free surface $x_2 = 0$, we have

$$\mathbf{u}'(x_1, 0, t) = \mathrm{i}\mathbf{A}\langle p_*\rangle \mathbf{q} e^{\mathrm{i}k(x_1 - vt)}, \tag{4.13}$$

where the prime denotes differentiation with kx_2. A linear superposition of $(3.2)_1$ for $p = p_1$, p_2, p_3 is

$$\mathbf{N}_1\mathbf{A}\mathbf{q} + \mathbf{N}_2\mathbf{B}\mathbf{q} = \mathbf{A}\langle p_*\rangle \mathbf{q}, \tag{4.14}$$

or, in view of (2.13) and $(3.1)_2$,

$$\mathbf{A}\langle p_*\rangle \mathbf{q} = \mathbf{N}_1\mathbf{a}_R. \tag{4.15}$$

Hence (4.13) has the expression

$$\mathbf{u}'(x_1, 0, t) = \mathbf{a}'_R e^{\mathrm{i}k(x_1 - vt)}, \quad \mathbf{a}'_R = \mathrm{i}\mathbf{N}_1\mathbf{a}_R. \tag{4.16}$$

Let

$$\mathbf{a}'_R = \mathbf{e}'_1 + \mathrm{i}\mathbf{e}'_2, \tag{4.17}$$

where $\mathbf{e}'_1$, $\mathbf{e}'_2$ are real. Eq. $(4.16)_2$ tells us that

$$\mathbf{e}'_1 = -\mathbf{N}_1\mathbf{e}_2, \quad \mathbf{e}'_2 = \mathbf{N}_1\mathbf{e}_1. \tag{4.18}$$

The differentiations of the square of the length of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$(\mathbf{e}_1 \cdot \mathbf{e}_1)' = 2\mathbf{e}_1 \cdot \mathbf{e}'_1, \quad (\mathbf{e}_2 \cdot \mathbf{e}_2)' = 2\mathbf{e}_2 \cdot \mathbf{e}'_2. \tag{4.19}$$

Eq. (4.19) provides information on whether the conjugate radii $\mathbf{e}_1$ and $\mathbf{e}_2$ increase their lengths as x_2 increases.

To study if the polarization plane rotates as x_2 increases, consider the vector

$$\mathbf{n} = (\mathbf{e}_1 \times \mathbf{e}_2), \tag{4.20}$$

which is normal to the polarization plane. The derivative of $\mathbf{n}$ is

$$\mathbf{n}' = (\mathbf{e}_1 \times \mathbf{e}'_2 + \mathbf{e}'_1 \times \mathbf{e}_2). \tag{4.21}$$

If $\mathbf{n}'$ is co-directional with $\mathbf{n}$, i.e.,

$$\mathbf{e}_1 \cdot \mathbf{n}' = 0, \quad \mathbf{e}_2 \cdot \mathbf{n}' = 0, \tag{4.22}$$

the polarization plane does not rotate. Hence when

$$|\mathbf{e}_1, \mathbf{e}_1', \mathbf{e}_2| = 0, \quad |\mathbf{e}_2, \mathbf{e}_2', \mathbf{e}_1| = 0, \tag{4.23}$$

the polarization plane does not rotate as x_2 increases. Otherwise it rotates.

5. Orthotropic materials

When the material is orthotropic, the reduced elastic compliances $s'_{\alpha\beta}$ vanish for $\alpha, \beta = 4, 5, 6$ except s'_{44}, s'_{55}, s'_{66}. The anti-plane and in-plane displacements are uncoupled so that only the in-plane displacement need to be considered. The polarization vector $\mathbf{a}_R$ is a two-vector and the matrices $\mathbf{K}^{(n)}$ are 2×2 matrices. Explicit expressions of $\mathbf{K}^{(n)}$ for $n = 1, 2, -1$, can be obtained by specializing the $\mathbf{K}^{(n)}$ for monoclinic materials shown in Appendix A. The nonzero elements are

$$\begin{aligned} K_{11}^{(1)} &= -(1 - s'_{11}X)/s'_{11}, \\ K_{22}^{(1)} &= X, \\ K_{12}^{(2)} &= [1 - (s'_{11} - s'_{12})X]/s'_{11}, \\ K_{11}^{(-1)} &= -X/(1 - s'_{66}X), \\ K_{22}^{(-1)} &= (1 - s'_{11}X)/[s'_{22} - s'(1,2)X]. \end{aligned} \tag{5.1}$$

Eq. (4.4) for $n = 1, 2, -1$ are

$$\begin{aligned} &K_{11}^{(1)} + X(\alpha_1^2 + \alpha_2^2) = 0, \\ &K_{12}^{(2)}\alpha_1 = 0, \\ &K_{11}^{(-1)} + K_{22}^{(-1)}(\alpha_1^2 + \alpha_2^2) = 0. \end{aligned} \tag{5.2}$$

Since $K_{12}^{(2)} \neq 0$, $(5.2)_2$ tells us that $\alpha_1 = 0$. Hence,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix}. \tag{5.3a}$$

Eq. $(5.2)_1$ gives, noticing that $\alpha_2 < 0$ from (4.6),

$$\alpha_2 = -\sqrt{(1 - s'_{11}X)/(s'_{11}X)} = -\sqrt{(C_{22} - X)/(C_{22}X)}. \tag{5.3b}$$

The second equality in (5.3b) is obtained by using the transformation rules between $s'_{\alpha\beta}$ and $C_{\alpha\beta}$ (Ting, 1997). Elimination of $(\alpha_1^2 + \alpha_2^2)$ between $(5.2)_{1,3}$ yields

$$XK_{11}^{(-1)} - K_{11}^{(1)}K_{22}^{(-1)} = 0, \tag{5.4a}$$

or

$$(1 - s'_{11}X)^2(1 - s'_{66}X) - s'_{11}[s'_{22} - s'(1,2)X]X^2 = 0. \tag{5.4b}$$

This is the secular equation in terms of $s'_{\alpha\beta}$. The secular equation in terms of $C_{\alpha\beta}$ was derived by Sveklo (1948) and Stoneley (1963).

The matrix $\mathbf{N}_1$, which is a 2×2 matrix here, can be obtained from (2.18), (2.19) as

$$\mathbf{N}_1 = \begin{bmatrix} 0 & -1 \\ s'_{12}/s'_{11} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -C_{12}/C_{22} & 0 \end{bmatrix}. \tag{5.5}$$

Eqs. (4.18) and (5.3a) give

$$\mathbf{e}_1' = \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2' = \begin{bmatrix} 0 \\ s_{12}'/s_{11}' \end{bmatrix}. \tag{5.6}$$

The conjugate radii $\mathbf{e}_1, \mathbf{e}_2$ shown in (5.3a) are orthogonal to each other so that $\mathbf{e}_1, \mathbf{e}_2$ are the principal axes of the ellipse. Eqs. (5.3a) and (5.6) tell us that the principal axes are fixed along the x_1- and x_2-axes. From (4.19) we also have

$$(\mathbf{e}_1 \cdot \mathbf{e}_1)' = 2\alpha_2, \quad (\mathbf{e}_2 \cdot \mathbf{e}_2)' = 2\alpha_2 s_{12}'/s_{11}'. \tag{5.7}$$

Since $\alpha_2 < 0$, the principal axis $\mathbf{e}_1$ decreases its length as x_2 increases. As to the principal axis $\mathbf{e}_2$, it increases (or decreases) its length when $s_{12}' < 0$ (or $s_{12}' > 0$). In the special case of isotropic materials, $s_{12}' = -\nu/(2\mu)$ where μ and ν are the shear modulus and Poisson ratio, respectively. The Poisson ratio ν is often assumed positive so that the principal axis $\mathbf{e}_2$ increases its length, as stated in the literature. Theoretically, $- < \nu < 1/2$ is sufficient to insure the positivity of strain energy density. Thus ν can be negative for isotropic materials, and $\mathbf{e}_2$ can decrease its length as x_2 increases.

6. Monoclinic materials with the symmetry plane at $x_3 = 0$

When the material has a symmetry plane at $x_3 = 0$,

$$s_{14}' = s_{24}' = s_{15}' = s_{25}' = s_{46}' = s_{56}' = 0. \tag{6.1}$$

Again we need to consider only the in-plane deformation so that $\mathbf{a}_R$ is a two-vector and $\mathbf{K}^{(n)}$ are 2×2 matrices. Explicit expressions of $\mathbf{K}^{(n)}$ for $n = 1, 2, -1$, are shown in Appendix A. Eq. (4.4) for $n = 1, 2, -1$ are

$$\begin{aligned} &K_{11}^{(1)} + X(\alpha_1^2 + \alpha_2^2) = 0, \\ &K_{11}^{(2)} + 2K_{12}^{(2)}\alpha_1 = 0, \\ &K_{11}^{(-1)} + K_{22}^{(-1)}(\alpha_1^2 + \alpha_2^2) + 2K_{12}^{(-1)}\alpha_1 = 0. \end{aligned} \tag{6.2}$$

Eqs. $(6.2)_{2,1}$ give

$$2\alpha_1 = -K_{11}^{(2)}/K_{12}^{(2)}, \quad \alpha_1^2 + \alpha_2^2 = -K_{11}^{(1)}/X, \tag{6.3}$$

or

$$\alpha_1 = \frac{s_{16}'(1 - s_{11}'X)}{s_{11}'[1 - (s_{11}' - s_{12}')X]}, \quad \alpha_2 = -\sqrt{[(1 - s_{11}'X)/(s_{11}'X)] - \alpha_1^2}. \tag{6.4a}$$

The conjugate radii are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ \alpha_1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix}. \tag{6.4b}$$

The principal radii are located at, from (4.11),

$$\tan 2\psi = \frac{2\alpha_1\alpha_2}{\alpha_2^2 - (1 + \alpha_1^2)}. \tag{6.5}$$

The $\mathbf{e}_1, \mathbf{e}_2$ in (6.4) are not the principal radii unless $\alpha_1 = 0$, i.e., $s_{16}' = 0$. It should be noted that $\mathbf{e}_1, \mathbf{e}_2$ in (6.4) are identical to $\mathbf{e}_1, \mathbf{e}_2$ in (5.3) for orthotropic materials when $s_{16}' = 0$ but s_{26}' need not vanish.

When (6.3) is inserted into $(6.2)_3$ we have

$$K_{12}^{(2)}[XK_{11}^{(-1)} - K_{11}^{(1)}K_{22}^{(-1)}] - XK_{11}^{(2)}K_{12}^{(-1)} = 0, \tag{6.6a}$$

or

$$[1-(s'_{11}-s'_{12})X]\left\{(1-s'_{11}X)^2(1-s'_{66}X) - s'_{11}[s'_{22}-s'(1,2)X]X^2\right\} \\ - s'_{16}(1-s'_{11}X)[(s'_{16}-2s'_{26})(1-s'_{11}X) - s'_{12}s'_{16}X]X^2 = 0. \tag{6.6b}$$

This is the most explicit secular equation obtained in Ting (2002c). Less explicit expressions were derived by Currie (1979), Destrade (2001) and Ting (2002a). In the special case of orthotropic materials, $s'_{16} = s'_{26} = 0$ so that, by inspection, (6.6b) simplifies to the secular equation in (5.4b) because $[1-(s'_{11}-s'_{12})X] \neq 0$. In reducing (6.6b) to (5.4b), s'_{26} need not vanish. Thus, when $s'_{16} = 0$, the secular equation for monoclinic materials with the symmetry plane at $x_3 = 0$ is identical to that for orthotropic materials. Another special case is when $s'_{12} = 0$. The secular equation (6.6b) is a product of $(1-s'_{11}X)^2$ and a quadratic equation in X (Ting, 2004). The smaller root of the quadratic equation is

$$X^{-1} = \frac{1}{2}(s'_{11}+s'_{66}) + \frac{1}{2}\sqrt{(s'_{11}-s'_{66})^2 + 4s'_{11}s'_{22} + 4s'_{16}(s'_{16}-2s'_{26})}. \tag{6.7a}$$

Eq. (6.4a) simplifies to

$$\alpha_1 = s'_{16}/s'_{11}, \quad \alpha_2 = -\sqrt{(s'_{11}X)^{-1} - (1+\alpha_1^2)}. \tag{6.7b}$$

Thus, when $s'_{12} = 0$, we have the exact expressions of X and the polarization vector $\mathbf{a}_R$. Other special cases for which an exact X can be obtained are presented in Ting (2004).

The matrix $\mathbf{N}_1$ is

$$\mathbf{N}_1 = \begin{bmatrix} \alpha_1 & -1 \\ s'_{12}/s'_{11} & 0 \end{bmatrix}. \tag{6.8}$$

Eqs. (4.18) and (4.19) reproduce (5.6) and (5.7) which are for orthotropic materials. Thus, as x_2 increases, $\mathbf{e}_1$ decreases its length while $\mathbf{e}_1$ increases (or decreases) its length when $s'_{12} < 0$ (or $s'_{12} > 0$).

7. Monoclinic materials with the symmetry plane at $x_1 = 0$

When the material has a symmetry plane at $x_1 = 0$,

$$s'_{15} = s'_{25} = s'_{16} = s'_{26} = s'_{45} = s'_{46} = 0. \tag{7.1}$$

The symmetric matrices $\mathbf{K}^{(n)}$ have the structure

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}. \tag{7.2}$$

The first of (7.2) applies to $\mathbf{K}^{(n)}$, $n = 1, 3, -1$, while the second of (7.2) applies to $n = 2, -2$. Explicit expressions of the $*$ elements of $\mathbf{K}^{(n)}$ are given in Appendix A.

Eq. (4.14) for $n = 2, -2$ simplifies to

$$\begin{aligned} K_{12}^{(2)}\alpha_1 + K_{13}^{(2)}\beta_1 &= 0, \\ K_{12}^{(-2)}\alpha_1 + K_{13}^{(-2)}\beta_1 &= 0. \end{aligned} \tag{7.3}$$

This is satisfied if $\alpha_1 = \beta_1 = 0$. We show in Appendix B that this is indeed the case. Hence the conjugate radii are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ \alpha_2 \\ \beta_2 \end{bmatrix}. \tag{7.4}$$

For $n = 1, -1, 3$, (4.4) has the expression

$$\begin{aligned} &K_{11}^{(1)} + X\alpha_2^2 + K_{33}^{(1)}\beta_2^2 = 0, \\ &K_{11}^{(-1)} + K_{22}^{(-1)}\alpha_2^2 + K_{33}^{(-1)}\beta_2^2 + K_{23}^{(-1)}(2\alpha_2\beta_2) = 0, \\ &K_{11}^{(3)} + K_{22}^{(3)}\alpha_2^2 + K_{33}^{(3)}\beta_2^2 + K_{23}^{(3)}(2\alpha_2\beta_2) = 0. \end{aligned} \tag{7.5}$$

This is rewritten as

$$\mathbf{k}_2\alpha_2^2 + \mathbf{k}_3\beta_2^2 + \mathbf{k}_4(2\alpha_2\beta_2) = -\mathbf{k}_1, \tag{7.6}$$

where

$$\mathbf{k}_1 = \begin{bmatrix} K_{11}^{(1)} \\ K_{11}^{(-1)} \\ K_{11}^{(3)} \end{bmatrix}, \quad \mathbf{k}_2 = \begin{bmatrix} X \\ K_{22}^{(-1)} \\ K_{22}^{(3)} \end{bmatrix}, \quad \mathbf{k}_3 = \begin{bmatrix} K_{33}^{(1)} \\ K_{33}^{(-1)} \\ K_{33}^{(3)} \end{bmatrix}, \quad \mathbf{k}_4 = \begin{bmatrix} 0 \\ K_{23}^{(-1)} \\ K_{23}^{(3)} \end{bmatrix}. \tag{7.7}$$

From (7.6) we obtain

$$\alpha_2 = -\sqrt{\frac{-|\mathbf{k}_1,\ \mathbf{k}_3,\ \mathbf{k}_4|}{|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_4|}}, \quad \beta_2 = \frac{-|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_1|}{2\alpha_2|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_4|}, \tag{7.8a}$$

and

$$\beta_2^2 = \frac{-|\mathbf{k}_2,\ \mathbf{k}_1,\ \mathbf{k}_4|}{|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_4|}. \tag{7.8b}$$

Elimination of α_2, β_2 among the three equations in (7.8) yields

$$|\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3|^2 + 4|\mathbf{k}_1,\ \mathbf{k}_3,\ \mathbf{k}_4| \cdot |\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_4| = 0. \tag{7.9}$$

This is a new secular equation which is different from the secular equations obtained in Ting (2002a,b). From the relation

$$|\mathbf{k}_i,\ \mathbf{k}_j,\ \mathbf{k}_n| = (\mathbf{k}_i \times \mathbf{k}_j)^{\mathrm{T}}\mathbf{k}_n = \mathbf{k}_n^{\mathrm{T}}(\mathbf{k}_i \times \mathbf{k}_j), \tag{7.10}$$

where the vector product $(\mathbf{k}_i \times \mathbf{k}_j)$ is a column matrix, (7.9) can be rewritten as

$$(\mathbf{k}_1 \times \mathbf{k}_3)^{\mathrm{T}}(4\mathbf{k}_4\mathbf{k}_4^{\mathrm{T}} - \mathbf{k}_2\mathbf{k}_3^{\mathrm{T}})(\mathbf{k}_1 \times \mathbf{k}_2) = 0. \tag{7.11}$$

In the special case of orthotropic materials, $\mathbf{k}_4 = \mathbf{0}$ so that (7.9) reduces to

$$|\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3| = 0. \tag{7.12}$$

The element $K_{11}^{(3)}$ in Appendix A simplifies to, for orthotropic materials,

$$s_{11}'^2 K_{11}^{(3)} = s_{12}'[2 - (2s_{11}' - s_{12}')X] + s_{66}'(1 - s_{11}'X)^2. \tag{7.13}$$

It can then be shown that, when (5.4b) holds, $\mathbf{k}_1$ and $\mathbf{k}_2$ are proportional to each other. Hence (7.12) holds for orthotropic materials. It is a challenge to show that $\beta_2 = 0$ and α_2 reduces to (5.3b) for orthotropic materials. This is another example that shows that special cases are best treated separately.

The conjugate radii $\mathbf{e}_1$, $\mathbf{e}_2$ shown in (7.4) are orthogonal to each other so that they are the principal radii. $\mathbf{e}_1$ is along the x_1-axis while $\mathbf{e}_2$ is on the plane $x_1 = 0$. The matrix $\mathbf{N}_1$ computed from (2.18) is

$$\mathbf{N}_1 = \begin{bmatrix} 0 & -1 & s_6 \\ r_2 & 0 & 0 \\ r_4 & 0 & 0 \end{bmatrix}, \tag{7.14a}$$

where

$$r_2 = s'_{12}/s'_{11}, \quad r_4 = s'_{14}/s'_{11}, \quad s_6 = s'_{56}/s'_{55}. \tag{7.14b}$$

Eqs. (4.18) and (4.19) give

$$\mathbf{e}'_1 = \begin{bmatrix} \alpha_2 - \beta_2 s_6 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}'_2 = \begin{bmatrix} 0 \\ r_2 \\ r_4 \end{bmatrix}, \tag{7.15}$$

$$(\mathbf{e}_1 \cdot \mathbf{e}_1)' = 2(\alpha_2 - \beta_2 s_6), \quad (\mathbf{e}_2 \cdot \mathbf{e}_2)' = 2(\alpha_2 r_2 + \beta_2 r_4). \tag{7.16}$$

In conjunction with (7.4), they tell us that the principal axis $\mathbf{e}_1$ remains along the x_1-axis as x_2 increases, and increases (or decreases) its length if $\alpha_2 - \beta_2 s_6 > 0$ (or <0). The principal axis $\mathbf{e}_2$ stays on the plane $x_1 = 0$ as x_2 increases. It increases (or decreases) its length if $\alpha_2 r_2 + \beta_2 r_4 > 0$ (or <0). Eq. $(4.23)_1$ is satisfied while $(4.23)_2$ is

$$|\mathbf{e}_2, \ \mathbf{e}'_2, \ \mathbf{e}_1| = \kappa, \quad \kappa = \alpha_2 r_4 - \beta_2 r_2. \tag{7.17}$$

If $\kappa = 0$, the vector $\mathbf{e}_2$ does not rotate as x_2 increases. Otherwise it rotates about the $\mathbf{e}_1$-axis counterclockwise (or clockwise) when $\kappa > 0$ (or <0).

8. Monoclinic materials with the symmetry plane at $x_2 = 0$

When the material has a symmetry plane at $x_2 = 0$,

$$s'_{14} = s'_{24} = s'_{16} = s'_{26} = s'_{45} = s'_{56} = 0. \tag{8.1}$$

The symmetric matrices $\mathbf{K}^{(n)}$ have the structure

$$\begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{bmatrix}. \tag{8.2}$$

The first of (8.2) applies to $\mathbf{K}^{(n)}$, $n = 1, 3, -1$, while the second of (8.2) applies to $n = 2, -2$. Explicit expressions of the $*$ elements of $\mathbf{K}^{(n)}$ are given in Appendix A.

Eq. (4.4) for $n = 2, -2$ simplifies to

$$\begin{aligned} &K^{(2)}_{12}\alpha_1 + K^{(2)}_{23}(\alpha_1\beta_1 + \alpha_2\beta_2) = 0, \\ &K^{(-2)}_{12}\alpha_1 + K^{(-2)}_{23}(\alpha_1\beta_1 + \alpha_2\beta_2) = 0. \end{aligned} \tag{8.3}$$

This is satisfied if $\alpha_1 = \beta_2 = 0$. We show in Appendix B that this is indeed the case. Hence the conjugate radii are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \beta_1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ \alpha_2 \\ 0 \end{bmatrix}. \tag{8.4}$$

For $n = 1, 3, -1$, (4.4) has the expression

$$\begin{aligned}
&K_{11}^{(1)} + X\alpha_2^2 + K_{33}^{(1)}\beta_1^2 + K_{13}^{(1)}(2\beta_1) = 0,\\
&K_{11}^{(-1)} + K_{22}^{(-1)}\alpha_2^2 + K_{33}^{(-1)}\beta_1^2 + K_{13}^{(-1)}(2\beta_1) = 0,\\
&K_{11}^{(3)} + K_{22}^{(3)}\alpha_2^2 + K_{33}^{(3)}\beta_1^2 + K_{13}^{(3)}(2\beta_1) = 0.
\end{aligned} \tag{8.5}$$

This is rewritten as

$$\mathbf{k}_2\alpha_2^2 + \mathbf{k}_3\beta_1^2 + \mathbf{k}_5(2\beta_1) = -\mathbf{k}_1, \tag{8.6}$$

where

$$\mathbf{k}_1 = \begin{bmatrix} K_{11}^{(1)} \\ K_{11}^{(-1)} \\ K_{11}^{(3)} \end{bmatrix}, \quad \mathbf{k}_2 = \begin{bmatrix} X \\ K_{22}^{(-1)} \\ K_{22}^{(3)} \end{bmatrix}, \quad \mathbf{k}_3 = \begin{bmatrix} K_{33}^{(1)} \\ K_{33}^{(-1)} \\ K_{33}^{(3)} \end{bmatrix}, \quad \mathbf{k}_5 = \begin{bmatrix} K_{13}^{(1)} \\ K_{13}^{(-1)} \\ K_{13}^{(3)} \end{bmatrix}. \tag{8.7}$$

From (8.6) we obtain

$$\alpha_2 = -\sqrt{\frac{-|\mathbf{k}_1,\ \mathbf{k}_3,\ \mathbf{k}_5|}{|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_5|}}, \quad \beta_1 = \frac{-|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_1|}{2|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_5|}, \tag{8.8a}$$

and

$$\beta_1^2 = \frac{-|\mathbf{k}_2,\ \mathbf{k}_1,\ \mathbf{k}_5|}{|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_5|}. \tag{8.8b}$$

Substitution of β_1 from (8.8a) into (8.8b) yields

$$|\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3|^2 + 4|\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_5| \cdot |\mathbf{k}_2,\ \mathbf{k}_1,\ \mathbf{k}_5| = 0. \tag{8.9}$$

This is a new secular equation which is different from the secular equations obtained in Ting (2002a,b). In the special case of orthotropic materials, $\mathbf{k}_5 = \mathbf{0}$ so that (8.9) reduces to (7.12) which, as stated there, holds when (5.4b) holds. Using (7.10), (8.9) can be rewritten as

$$(\mathbf{k}_2 \times \mathbf{k}_3)^{\mathrm{T}}(4\mathbf{k}_5\mathbf{k}_5^{\mathrm{T}} - \mathbf{k}_1\mathbf{k}_3^{\mathrm{T}})(\mathbf{k}_2 \times \mathbf{k}_1) = 0. \tag{8.10}$$

The conjugate radii $\mathbf{e}_1$, $\mathbf{e}_2$ shown in (8.4) are orthogonal to each other so that they are the principal radii. $\mathbf{e}_1$ is on the plane $x_2 = 0$ while $\mathbf{e}_2$ is along the negative x_2-axis. The matrix $\mathbf{N}_1$ computed from (2.18) is

$$\mathbf{N}_1 = \begin{bmatrix} 0 & -1 & 0 \\ r_2 & 0 & s_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.11a}$$

where

$$r_2 = \frac{s'(1,5|2,5)}{s'(1,5)}, \quad s_2 = \frac{s'(1,5|1,2)}{s'(1,5)}. \tag{8.11b}$$

Eqs. (4.18) and (4.19) give

$$\mathbf{e}_1' = \begin{bmatrix} \alpha_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2' = \begin{bmatrix} 0 \\ r_2 + \beta_1 s_2 \\ 0 \end{bmatrix}, \tag{8.12}$$

$$(\mathbf{e}_1 \cdot \mathbf{e}_1)' = 2\alpha_2, \quad (\mathbf{e}_2 \cdot \mathbf{e}_2)' = 2\alpha_2(r_2 + \beta_1 s_2). \tag{8.13}$$

In conjunction with (8.4), they tell us that the principal axis $\mathbf{e}_1$ stays on the plane $x_1 = 0$ as x_2 increases, and decreases its length since $\alpha_2 < 0$. The principal axis $\mathbf{e}_2$ remains along the x_2-axis as x_2 increases. It increases (or decreases) its length if $r_2 + \beta_1 s_2 < 0$ (or >0). Eq. $(4.23)_1$ is satisfied while $(4.23)_2$ is

$$|\mathbf{e}_1,\ \mathbf{e}_1',\ \mathbf{e}_2| = \beta_1\alpha_2^2. \tag{8.14}$$

If $\beta_1 = 0$, $\mathbf{e}_1$ does not rotate as x_2 increases. Otherwise, it rotates about the $\mathbf{e}_2$-axis counter-clockwise (or clockwise) when $\beta_1 > 0$ (or <0).

9. Two- and one-component surface waves

The analysis presented in Sections 3 and 4 assumed that the surface wave consists of three partial waves so that all three components of the constant vector $\mathbf{q}$ in (2.13) are nonzero. For a two-component surface wave, we can set $q_3 = 0$ so that $\mathbf{q}$ is a two-vector. The vectors $\mathbf{a}_3$ and $\mathbf{b}_3$ are not needed. Hence the matrices $\mathbf{A}$ and $\mathbf{B}$ consist of two columns, and are 3×2 matrices. With these modifications, all derivations in Sections 3 and 4 remain valid for two-component surface waves.

In the case of one-component surface waves, we may set $q_1 = 1$ and $q_2 = q_3 = 0$. Thus $\mathbf{q}$ is replaced by a scalar of unity while $\mathbf{A}$ and $\mathbf{B}$ are replaced by single vectors $\mathbf{a}$ and $\mathbf{b}$. The polarization vector $\mathbf{a}_R$ is simply $\mathbf{a}$. For a one-component surface wave it is known that (Barnett and Chadwick, 1990; see also Ting, 1996, Section 12.8)

$$s'_{12} = s'_{15} = s'_{25} = 0, \quad s'_{11} = s'_{55} = X^{-1}. \tag{9.1}$$

It can then be shown that $\mathbf{K}^{(n)}$ vanishes for $n = 2, 4, -2$. For $n = 1, 3, 5, -1$, $K_{22}^{(n)}$ is the only nonzero element of $\mathbf{K}^{(n)}$. In fact

$$K_{22}^{(1)} = 1/s'_{11}, \quad K_{22}^{(3)} = s'_{22}/s'^2_{11}, \quad K_{22}^{(5)} = s'^2_{22}/s'^3_{11}. \tag{9.2}$$

Thus (4.4) is trivially satisfied for $n = 2, 4, -2$, and

$$K_{22}^{(n)}(\alpha_1^2 + \alpha_2^2) = 0, \tag{9.3}$$

for $n = 1, 3, 5, -1$. In view of (9.2), (9.3) gives $\alpha_1 = \alpha_2 = 0$. Thus the polarization vector is on the plane $x_2 = 0$, confirming what is known in the literature. Unfortunately, the present analysis do not determine β_1 and β_2.

10. Concluding remarks

The secular equation due to Taziev (1989) for a general anisotropic elastic material is derived using a different derivation. In the process, an explicit expression of the polarization vector is obtained. The same derivation is employed to obtain the secular equations and the polarization vectors $\mathbf{e}_1 + i\mathbf{e}_2$ for orthotropic and monoclinic materials with the symmetry plane at $x_3 = 0$, $x_1 = 0$ or $x_2 = 0$. The displacement at the free surface traces an elliptic path for which $\mathbf{e}_1$, $\mathbf{e}_2$ are a pair of conjugate radii of the ellipse. New secular equations are obtained for monoclinic materials with the symmetry plane at $x_1 = 0$ or $x_2 = 0$. The vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are explicit in terms of the elastic compliances and $X = \rho v^2$ where ρ and v are the mass density and surface wave speed, respectively. They will be useful as test examples for a numerical solution of surface waves. For monoclinic materials with the symmetry plane at $x_1 = 0$ (or $x_2 = 0$), the polarization plane contains the x_1-axis (or the x_2-axis) which is a principal axis of the ellipse. We also study if $\mathbf{e}_1$, $\mathbf{e}_2$ and the polarization plane change as we moves away from the free surface.

Appendix A. The matrix $\mathbf{K}^{(n)}$ for monoclinic materials

The elements of the 3×3 symmetric matrices $\mathbf{K}^{(n)}$ can be computed using the recursive formula (3.12). $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(-1)}$ are obtained from (3.8) and (3.9) in which the quantities needed in the equations can be found in (2.7b), (2.18) and (2.19). The expressions in terms of $s'_{\alpha\beta}$ and its minors can be converted to $C_{\alpha\beta}$ and its minors, and vice versa (Ting, 1997). Listed below are $\mathbf{K}^{(n)}$ for monoclinic materials with the symmetry plane at $x_3=0$, $x_1=0$, and $x_2=0$. As pointed out below, explicit expressions of some of the elements of $\mathbf{K}^{(n)}$ are not needed in the paper.

A.1. Symmetry plane at $x_3 = 0$

The $s'_{\alpha\beta}$ shown in (6.1) vanish for this case. Since the in-plane and the anti-plane deformations are uncoupled, it suffices to consider the in-plane deformation. The $\mathbf{K}^{(n)}$ are reduced to 2×2 matrices so that only $K_{11}^{(n)}$, $K_{22}^{(n)}$ and $K_{12}^{(n)}$ are shown below.

$$K_{11}^{(1)} = X - (1/s'_{11}), \quad K_{22}^{(1)} = X, \quad K_{12}^{(1)} = 0,$$

$$K_{11}^{(2)} = -2s'_{16}(1 - s'_{11}X)/s'^2_{11}, \quad K_{22}^{(2)} = 0,$$
$$K_{12}^{(2)}[1 - (s'_{11} - s'_{12})X]/s'_{11},$$

$$\begin{aligned}
K_{11}^{(-1)} &= -[C(1,6) - C_{66}X]X/D_3 = -[s'_{22} - s'(1,2)X]X/D'_3,\\
K_{22}^{(-1)} &= \{C(1,2,6) - [C(1,2) + C(2,6)]X + C_{22}X^2\}/D_3\\
&= [(1 - s'_{11}X)(1 - s'_{66}X) - (s'_{16}X)^2]/D'_3,\\
K_{12}^{(-1)} &= -[C(1,2|1,6) - C_{26}X]X/D_3 = [s'_{26} - s'(1,2|1,6)X]X/D'_3,
\end{aligned}$$

where

$$D_3 = (C_{11} - X)(C_{66} - X) - C_{16}^2,$$

$$\begin{aligned}
D'_3 &= s'_{22} - [s'(2,6) + s'(1,2)]X + s'(1,2,6)X^2\\
&= [s'_{22} - s'(1,2)X](1 - s'_{66}X) + s'_{16}s'(1,2|2,6)X^2 + s'_{26}[s'_{26} - s'(1,2|1,6)X]X.
\end{aligned}$$

It should be noted that D_3 or D'_3 is a common factor for $K_{ij}^{(-1)}$ so that it can be deleted in using $(6.2)_3$ and (6.6a).

A.2. Symmetry plane at $x_1 = 0$

The $s'_{\alpha\beta}$ shown in (7.1) vanish for this case. The matrices $\mathbf{K}^{(n)}$ have the structure $(7.2)_1$ for $n = 1, 3, -1$ and the structure $(7.2)_2$ for $n = 2, -2$. Listed below are the $*$ elements of $\mathbf{K}^{(n)}$ shown in (7.2).

$$K_{11}^{(1)} = X - (1/s'_{11}), \quad K_{22}^{(1)} = X,$$
$$K_{33}^{(1)} = X - (1/s'_{55}), \quad K_{23}^{(1)} = 0,$$

$$s'_{11}K_{12}^{(2)} = 1 - (s'_{11} - s'_{12})X,$$

$$s'_{11}s'_{55}K_{13}^{(2)} = -s'_{14}(1 - s'_{55}X) - s'_{56}(1 - s'_{11}X),$$

$$s'_{55}s'^2_{11}K^{(3)}_{11} = s'_{55}s'^2_{12}X - s'^2_{14}(1 - s'_{55}X) - [2(s'_{14}s'_{56} - s'_{12}s'_{55}) - s'(5,6)(1 - s'_{11}X)](1 - s'_{11}X),$$

$$s'_{11}K^{(3)}_{22} = -1 + (s'_{11} - 2s'_{12})X + s'(1,2)X^2,$$

$$s'_{11}s'^2_{55}K^{(3)}_{33} = -s'^2_{56}(1 - s'_{11}X) - [2s'_{14}s'_{56} - s'(1,4)(1 - s'_{55}X)](1 - s'_{55}X),$$

$$s'_{11}s'_{55}K^{(3)}_{23} = s'_{56}[1 - (s'_{11} - s'_{12})X] + [s'_{14} - s'(1,2|1,4)X)](1 - s'_{55}X),$$

$$K^{(-1)}_{11} = -[C(5,6) - C_{66}X]X/D_1 = -(1 - s'_{55}X)X/D'_1,$$

$$K^{(-1)}_{22} = [C(1,2) - C_{22}X]/(C_{11} - X) = [s'_{44} - s'(1,4)X]/\widehat{D}_1,$$

$$K^{(-1)}_{33} = [C(1,4) - C_{44}X]/(C_{11} - X) = [s'_{22} - s'(1,2)X]/\widehat{D}_1,$$

$$K^{(-1)}_{23} = [C(1,2|1,4) - C_{24}X]/(C_{11} - X) = -[s'_{24} - s'(1,2|1,4)X]/\widehat{D}_1,$$

where

$$D_1 = (C_{55} - X)(C_{66} - X) - C^2_{56},$$

$$D'_1 = (1 - s'_{55}X)(1 - s'_{66}X) - (s'_{56}X)^2,$$

$$\widehat{D}_1 = s'(2,4) - s'(1,2,4)X.$$

It should be noted that explicit expressions of $\mathbf{K}^{(2)}$ and $\mathbf{K}^{(-2)}$ are not needed in (7.3) because α_1 and β_1 in (7.3) vanish. However, we need $\mathbf{K}^{(2)}$ to compute $\mathbf{K}^{(3)}$.

A.3. Symmetry plane at $x_2 = 0$

The $s'_{\alpha\beta}$ shown in (8.1) vanish for this case. The matrices $\mathbf{K}^{(n)}$ have the structure $(8.2)_1$ for $n = 1, 3, -1$ and the structure $(8.2)_2$ for $n = 2, -2$. Listed below are the $*$ elements of $\mathbf{K}^{(n)}$ shown in (8.2).

$$K^{(1)}_{11} = X - [s'_{55}/s'(1,5)], \quad K^{(1)}_{22} = X,$$

$$K^{(1)}_{33} = X - [s'_{11}/s'(1,5)], \quad K^{(1)}_{13} = s'_{15}/s'(1,5),$$

$$s'(1,5)K^{(2)}_{12} = s'_{55} + [s'(1,5|2,5) - s'(1,5)]X,$$

$$s'(1,5)K^{(2)}_{23} = -s'_{15} + s'(1,2|1,5)X,$$

$$[s'(1,5)]^2K^{(3)}_{11} = s'_{44}s'^2_{15} + [s'(1,5|2,5)]^2X + \{2s'(1,5|2,5) - 2s'_{15}s'_{46} + s'_{66}[s'_{55} - s'(1,5)X]\}[s'_{55} - s'(1,5)X],$$

$$s'(1,5)K^{(3)}_{22} = -s'_{55} + [s'(1,5) - 2s'(1,5|2,5)]X + s'(1,2,5)X^2,$$

$$[s'(1,5)]^2K^{(3)}_{33} = s'_{66}s'^2_{15} - s'(1,2|1,5)[2s'_{15} - s'(1,2|1,5)X] - \{2s'_{15}s'_{46} - s'_{44}[s'_{11} - s'(1,5)X]\}[s'_{11} - s'(1,5)X],$$

$$[s'(1,5)]^2K^{(3)}_{13} = -s'(1,5|2,5)[s'_{15} - s'(1,2|1,5)X] + s'_{15}\{s'_{15}s'_{46} - s'_{44}[s'_{11} - s'(1,5)X]\} + \{s'(1,2|1,5) - s'_{15}s'_{66} + s'_{46}[s'_{11} - s'(1,5)X]\}[s'_{55} - s'(1,5)X],$$

$$K_{11}^{(-1)} = -C_{66}X/(C_{66} - X) = -s'_{44}X/\widehat{D}_2,$$
$$K_{22}^{(-1)} = \{C(1,2,5) - [C(1,2) + C(2,5)]X + C_{22}X^2\}/D_2$$
$$= [(1 - s'_{11}X)(1 - s'_{55}X) - (s'_{15}X)^2]/D'_2,$$
$$K_{33}^{(-1)} = [C(4,6) - C_{44}X]/(C_{66} - X) = (1 - s'_{66}X)/\widehat{D}_2,$$
$$K_{13}^{(-1)} = -C_{46}X/(C_{66} - X) = s'_{46}X/\widehat{D}_2,$$

where

$$D_2 = (C_{11} - X)(C_{55} - X) - C_{15}^2,$$
$$D'_2 = s'_{22} - [s'(1,2) + s'(2,5)]X + s'(1,2,5)X^2,$$
$$\widehat{D}_2 = s'_{44} - s'(4,6)X.$$

Again, explicit expressions of $\mathbf{K}^{(2)}$ and $\mathbf{K}^{(-2)}$ are not needed in (8.3) because α_1 and β_2 in (8.3) vanish. However, we need $\mathbf{K}^{(2)}$ to compute $\mathbf{K}^{(3)}$.

Appendix B. Polarization planes for monoclinic materials

We will show that the polarization plane for monoclinic materials with the symmetry plane at $x_1 = 0$ (or $x_2 = 0$) contains the x_1-axis (or the x_2-axis) as shown in (7.4) (or (8.4)).

For monoclinic materials with the symmetry plane at $x_1 = 0$, the eigenrelation (2.4) and the vector $\mathbf{b}$ in $(2.8)_1$ have the expression

$$\Gamma\mathbf{a} = \begin{bmatrix} C_{11} + p^2C_{66} & p(C_{12} + C_{66}) & p(C_{14} + C_{56}) \\ p(C_{12} + C_{66}) & C_{66} + p^2C_{22} & C_{56} + p^2C_{24} \\ p(C_{14} + C_{56}) & C_{56} + p^2C_{24} & C_{55} + p^2C_{44} \end{bmatrix}\mathbf{a} = \mathbf{0}, \tag{B.1}$$

$$\mathbf{b} = \begin{bmatrix} pC_{66} & C_{66} & C_{56} \\ C_{12} & pC_{22} & pC_{24} \\ C_{14} & pC_{24} & pC_{44} \end{bmatrix}\mathbf{a}. \tag{B.2}$$

The sextic equation $|\Gamma| = 0$ is a cubic equation in p^2. If p_1^2, p_2^2, p_3^2 are all real, we have

Case I

$$p_1 = \mathrm{i}\gamma_1, \quad p_2 = \mathrm{i}\gamma_2, \quad p_3 = \mathrm{i}\gamma_3. \tag{B.3}$$

If p_1^2, p_2^2 are complex conjugates and p_3^2 is real, we have

Case II

$$p_1 = \gamma_1 + \mathrm{i}\gamma_2, \quad p_2 = -\gamma_1 + \mathrm{i}\gamma_2, \quad p_3 = \mathrm{i}\gamma_3. \tag{B.4}$$

In (B.3) and (B.4), γ_1, γ_2, γ_3 are real and positive.

Consider Case I first. The eigenvectors $\mathbf{a}_k$ $(k = 1, 2, 3)$ of (B.1) assume the structure

$$\mathbf{a}_k^{\mathrm{T}} = [R,\ I,\ I], \tag{B.5}$$

where R and I stand for *real* and *pure imaginary*, respectively. The vectors $\mathbf{b}_k$ computed from (B.2) have the structure

$$\mathbf{b}_k^{\mathrm{T}} = [I,\ R,\ R]. \tag{B.6}$$

The vanishing of the surface traction $\mathbf{Bq} = \mathbf{0}$ in (2.13) is satisfied by taking

$$\mathbf{q}^{\mathrm{T}} = [R,\ R,\ R]. \tag{B.7}$$

The polarization vector $\mathbf{a}_R = \mathbf{Aq}$ then gives

$$\mathbf{a}_R^{\mathrm{T}} = [R,\ I,\ I]. \tag{B.8}$$

This proves (7.4).

For Case II for which the eigenvalues p are given by (B.4),

$$p_2 = -\bar{p}_1 \quad \text{and} \quad p_3 = \mathrm{i}\gamma_3. \tag{B.9}$$

Let the eigenvector $\mathbf{a}_1$ associated with p_1 be given by

$$\mathbf{a}_1^{\mathrm{T}} = [\xi_1,\ \xi_2,\ \xi_3]. \tag{B.10a}$$

It can be shown from (B.1) that

$$\mathbf{a}_2^{\mathrm{T}} = [-\bar{\xi}_1,\ \bar{\xi}_2,\ \bar{\xi}_3], \tag{B.10b}$$

$$\mathbf{a}_3^{\mathrm{T}} = [R,\ I,\ I]. \tag{B.10c}$$

Likewise, if the vector $\mathbf{b}_1$ computed from (B.2) is

$$\mathbf{b}_1^{\mathrm{T}} = [\eta_1,\ \eta_2,\ \eta_3], \tag{B.11a}$$

we have

$$\mathbf{b}_2^{\mathrm{T}} = [\bar{\eta}_1,\ -\bar{\eta}_2,\ -\bar{\eta}_3], \tag{B.11b}$$

$$\mathbf{b}_3^{\mathrm{T}} = [I,\ R,\ R]. \tag{B.11c}$$

The vanishing of the surface traction $\mathbf{Bq} = \mathbf{0}$ in (2.13) is satisfied by taking

$$\mathbf{q}^{\mathrm{T}} = [\lambda,\ -\bar{\lambda},\ R]. \tag{B.12}$$

The polarization vector $\mathbf{a}_R = \mathbf{Aq}$ then gives

$$\mathbf{a}_R^{\mathrm{T}} = \begin{bmatrix} \xi_1 & -\bar{\xi}_1 & R \\ \xi_2 & \bar{\xi}_2 & I \\ \xi_3 & \bar{\xi}_3 & I \end{bmatrix} \begin{bmatrix} \lambda \\ -\bar{\lambda} \\ R \end{bmatrix} = \begin{bmatrix} \lambda\xi_1 + \bar{\lambda}\bar{\xi}_1 + R \\ \lambda\xi_2 - \bar{\lambda}\bar{\xi}_2 + I \\ \lambda\xi_3 - \bar{\lambda}\bar{\xi}_3 + I \end{bmatrix}, \tag{B.13}$$

which leads to (B.8). This completes the proof.

For monoclinic materials with the symmetry plane at $x_2 = 0$, (B.1) and (B.2) are replaced by

$$\Gamma\mathbf{a} = \begin{bmatrix} C_{11} + p^2C_{66} & p(C_{12} + C_{66}) & C_{15} + p^2C_{46} \\ p(C_{12} + C_{66}) & C_{66} + p^2C_{22} & p(C_{46} + C_{25}) \\ C_{15} + p^2C_{46} & p(C_{46} + C_{25}) & C_{55} + p^2C_{44} \end{bmatrix} \mathbf{a} = \mathbf{0}, \tag{B.14}$$

$$\mathbf{b} = \begin{bmatrix} pC_{66} & C_{66} & pC_{46} \\ C_{12} & pC_{22} & C_{25} \\ pC_{46} & C_{46} & pC_{44} \end{bmatrix} \mathbf{a}. \tag{B.15}$$

Again, $|\Gamma| = 0$ is a cubic equation in p^2 so that either (B.3) or (B.4) applies. For Case I, we have

$$\mathbf{a}_k^{\mathrm{T}} = [R,\ I,\ R] \quad (k = 1, 2, 3). \tag{B.16}$$

The vectors $\mathbf{b}_k$ computed from (B.15) have the structure

$$\mathbf{b}_k^{\mathrm{T}} = [I,\ R,\ I]. \tag{B.17}$$

The vanishing of the surface traction $\mathbf{Bq} = \mathbf{0}$ is satisfied by taking

$$\mathbf{q}^{\mathrm{T}} = [R,\ R,\ R]. \tag{B.18}$$

The polarization vector $\mathbf{a}_R = \mathbf{Aq}$ then gives

$$\mathbf{a}_R^{\mathrm{T}} = [R,\ I,\ R]. \tag{B.19}$$

This proves (8.4).

For Case II for which the eigenvalues p are given by (B.4) or (B.9), it can be shown that the polarization vector also has the structure (B.19). The proof is similar to the one for monoclinic materials with the symmetry plane at $x_1 = 0$.

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